Name ________________________ ID3 _ _ _

ECS 315: Probability and Random Processes **2019/1**

HW 3 — Due: September 12, 4 PM

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Instructions

(a) This assignment has 5 pages.

(b) (1 pt) Hard-copies are distributed in class. Original pdf file can be downloaded from the course website. Work and write your answers **directly on the provided hardcopy/file** (not on other blank sheet(s) of paper).

(c) (1 pt) Write your first name and the last three digits of your student ID in the spaces provided on the upper-right corner of this page.

(d) (8 pt) Try to solve all non-optional problems.

(e) Late submission will be heavily penalized.

Problem 1. If A, B, and C are disjoint events with $P(A) = 0.2$, $P(B) = 0.3$ and $P(C) = 0.4$, determine the following probabilities:

(a) $P(A \cup B \cup C)$

(b) $P(A \cap B \cap C)$

(c) $P(A \cap B)$

(d) $P\left((A \cup B) \cap C\right)$

(e) $P(A^c \cap B^c \cap C^c)$

[Montgomery and Runger, 2010, Q2-75]

Problem 2. The sample space of a random experiment is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

[Montgomery and Runger, 2010, Q2-55]

Problem 3. ***Binomial theorem***: For any positive integer n, we know that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{3.1}$$

(a) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

(b) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

☆ .... $\binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5} = 0$

(c) Use the binomial theorem (3.2) to evaluate $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$. $= \left((-1)+1\right)^n = 0$

$x = -1$

$y = 1$

$\sum_{k=0}^{n} \binom{n}{k} = 2^n$

☆ + *

$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} 2\binom{n}{k} = 2^n + 0$

* .... $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5$

Problem 4. Let A and B be events for which $P(A)$, $P(B)$, and $P(A \cup B)$ are known. Express the following probabilities in terms of the three known probabilities above.

(a) $P(A \cap B)$

(b) $P(A \cap B^c)$

(c) $P(B \cup (A \cap B^c))$

(d) $P(A^c \cap B^c)$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. ***Binomial theorem***: For any positive integer n, we know that

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{3.2}$$

(a) Use the binomial theorem (3.2) to simplify the following sums

(i) $\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

$(-x+y)^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r x^r y^{n-r}$

(ii) $\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

(b) If we differentiate (3.2) with respect to x and then multiply by x, we have

$$\sum_{r=0}^{n} r \binom{n}{r} x^r y^{n-r} = nx(x+y)^{n-1}.$$

Use similar technique to simplify the sum $\sum_{r=0}^{n} r^2 \binom{n}{r} x^r y^{n-r}$.

Problem 6.

(a) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Find the range of possible values for $P(A \cap B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.21]

(b) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Find the range of possible values for $P(A \cup B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.22]

Problem 7. (**Classical Probability and Combinatorics**) Suppose n integers are chosen with replacement (that is, the same integer could be chosen repeatedly) at random from $\{1, 2, 3, \ldots, N\}$. Calculate the probability that the chosen numbers arise according to some non-decreasing sequence.

5

10

$N^n = 10^5$

1 1 3 1 2

9 1 4 4 3

9,10,10,10,10

2,2,3,3,4

4,5,5,5,5

$x_2 = 2$
$x_3 = 2$
$x_4 = 1$

$x_j = 0$ for $j \neq 2, 3, 4$

| 1111 ||||||||

$$\underset{}{x_1} + \overset{1+1}{x_2} + \overset{1+1}{x_3} + \overset{1}{x_4} + x_5 + \cdots + x_{10} = 1+1+1+1+1$$

n

N

$$\binom{n+N-1}{n} = \frac{(n+N-1)!}{n!\,(N-1)!}$$

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$$\frac{14!}{5!\,9!}$$

$$(x_2 + x_3) = 20 - 2x_1 \qquad \frac{7!}{5!\,2!}$$

$x_1 \geq 0$

$x_2 \geq 0$

$x_3 \geq 0$

$\cancel{x_1 \geq 6}$

$\cancel{x_2 \geq 5}$

$\cancel{x_3 \geq 4}$

$$\sum_{x_1=0}^{10}$$

$$x_1 + x_2 + x_3 = 5$$

$$\frac{7!}{5!\,2!}$$

Find # solutions of

$$2x_1 + x_2 + x_3 = 20$$

Assume $x_i \geq 0$

$$x_2 + x_3 = \underbrace{20 - 2x_1}$$

$$\frac{(20 - 2x_1 + 1)!}{(20 - 2x_1)!\,1!} = 20 - 2x_1 + 1 = 21 - 2x_1$$

$$\sum_{x_1=0}^{10} (21 - 2x_1) = 21 \times 11 - 2\sum_{x_1=0}^{10} x_1$$

$$\cancel{0} + 1 + 2 + \cdots + 10 = A$$

$$10 + 9 + \cdots + 1 = A$$

$$11 \times 10 = 2A$$

$$A = 55$$